M252 2002 Paper.

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1ai An experiment is a process which leads to a series of outcomes.

1aii A sample space is a set of possible outcomes of an experiment

1aiii An event is a collection of outcomes.

1aiv A random variable is a rule for allocating numbers to outcomes.

1av A Bernoulli trial is an experiment which has only 2 possible outcomes, a "success" or a "failure".

1b. $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$P(B) \neq 0$.

1c. $P(A|C) = \dfrac{P(A \cap C)}{P(C)}$.

$P(B|C) = \dfrac{P(B \cap C)}{P(C)}$

$\therefore \dfrac{P(A|C)}{P(B|C)} = \dfrac{P(A \cap C)}{P(B \cap C)}$

1d. Let A be the event that an individual caught a flu.
Let B be the event that an individual did not catch a flu.
Let L be the event that an individual is given a low dosage
Let M be the event that an individual is given a medium dosage
Let H be the event that an individual is given a high dosage

$P(A|L) = \dfrac{P(A \cap L)}{P(L)} \cancel{\approx} \cancel{\dfrac{0.0024\ 0.024}{0.313}} \cancel{= 0.0767}$

$= \dfrac{3/125}{313/1000} = \dfrac{24}{313}$

$P(B|L) = \dfrac{289/1000}{313/1000} = \dfrac{289}{313}$

$$\therefore \quad \frac{P(A\cap L)}{P(B\cap L)} = \frac{P(A|L)}{P(B|L)} = \frac{24}{289}$$

Similarly, $\quad \frac{P(A\cap M)}{P(B\cap M)} = \frac{P(A|M)}{P(B|M)} = \frac{9}{100}$

and $\quad \frac{P(A\cap H)}{P(B\cap H)} = \frac{P(A|H)}{P(B|H)} = \frac{13}{565}$ $\square$

2ai Let X_1 be the number of defectives found

$X_1 \sim H(5, 3, 20)$

$$P(X_1 = 2) = \frac{\binom{3}{2}\binom{17}{3}}{\binom{20}{5}} = 0.132 \quad \square$$

2aii ~~7th~~ ~~compo~~ P(7th component inspected is 2nd defective)

= P(1 defective found in the first 6 components inspected) $\times$ P(7th component is defective)

Let X_2 be the number of defectives found

$X_2 \sim H(6, 3, 20)$.

$$P(X_2 = 1) = \frac{\binom{6}{1}\binom{\cancel{14}\,17}{5}}{\binom{20}{6}} = 0.95789.$$

P(7th component inspected is 2nd defective)

$$= P(X_2 = 1) \times \frac{2}{14} = 0.137 \quad \square$$

2aiii Let X_3 be the number of components that need to be inspected before finding all the defectives.

$X_3 \in \{3, 4, 5, \ldots, 19, 20\}$

$$P(X_3 \leq 10) = \frac{\binom{3}{3}}{\binom{20}{3}} + \frac{\binom{17}{1}}{\binom{20}{4}} + \frac{\binom{17}{2}}{\binom{20}{5}} + \cdots + \frac{\binom{17}{7}}{\binom{20}{10}} = \frac{11}{38} = 0.289 \quad \square$$

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26. $Y \in \{3, 4, 5, \ldots, 20\}$

$P(Y=3) = \dfrac{\binom{17}{0}}{\binom{20}{3}} = \dfrac{1}{1140}$

$P(Y=4) = \dfrac{17}{\binom{20}{4}} = \dfrac{1}{285}$

$P(Y=5) = \dfrac{\binom{17}{2}}{\binom{20}{5}} = \dfrac{1}{114}$

$P(Y=6) = \binom{17}{3} \Big/ \binom{20}{6} = \dfrac{1}{57}$

$P(Y=7) = \binom{17}{4} \Big/ \binom{20}{7} = \dfrac{7}{228}$

$P(Y=8) = \binom{17}{5} \Big/ \binom{20}{8} = \dfrac{14}{285}$

$P(Y=9) = \binom{17}{6} \Big/ \binom{20}{9} = \dfrac{7}{95}$

$P(Y=10) = \binom{17}{7} \Big/ \binom{20}{10} = \dfrac{2}{19}$

$P(Y=11) = \binom{17}{8} \Big/ \binom{20}{11} = \dfrac{11}{76}$

$P(Y=12) = \binom{17}{9} \Big/ \binom{20}{12} = \dfrac{11}{57}$

$P(Y=13) = \binom{17}{10} \Big/ \binom{20}{13} = \dfrac{143}{570}$

$P(Y=14) = \binom{17}{11} \Big/ \binom{20}{14} = \dfrac{91}{285}$

$P(Y=15) = \binom{17}{12} \Big/ \binom{20}{15} = \dfrac{91}{228}$

$P(Y=16) = \binom{17}{13} \Big/ \binom{20}{16} = \dfrac{28}{57}$

$P(Y=17) = \binom{17}{14} \Big/ \binom{20}{17} = \dfrac{34}{57}$

$P(Y=18) = \binom{17}{15} \Big/ \binom{20}{18} = \dfrac{68}{95}$

$P(Y=19) = \binom{17}{16} \Big/ \binom{20}{19} = \dfrac{17}{20}$

$P(Y=20) = \binom{17}{17} \Big/ \binom{20}{20} = 1$

$$P(Y=k) = \frac{\binom{17}{k-3}}{\binom{20}{k}} = \frac{17!}{(k-3)!\,(17-k+3)!} \div \frac{20!}{k!\,(20-k)!}$$

$$= \frac{17!}{(k-3)!\,(\cancel{20-k})!} \times \frac{k!\,(\cancel{20-k})!}{20!}$$

$$= \frac{1}{(18)(19)(20)}(k-1)(k-2)$$

$$= \frac{1}{6840}(k-1)(k-2)$$

$\therefore \; c = \dfrac{1}{6840}$ $\square$

Mode of Y = value which Y takes that has the highest probability

$= 20.$ $\square$

AZONE

3a $f(x) = \lambda e^{-\lambda x}$

$E(X) = \lambda \int_0^\infty x e^{-\lambda x}\, dx = \cancel{\lambda} \left[x(-e^{-\lambda x})\right]_0^\infty + \int_0^\infty e^{-\lambda x}\, dx$

$= \left[-\frac{1}{\lambda} e^{-\lambda x}\right]_0^\infty = \frac{1}{\lambda}$ $\square$

$E(X^2) = \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx = \cancel{\lambda} \int_0^\infty 2x e^{-\lambda x}\, dx = \left[2x\left(-\frac{e^{-\lambda x}}{\lambda}\right)\right]_0^\infty + \int_0^\infty 2 \frac{e^{-\lambda x}}{\lambda}\, dx$

$= \frac{2}{\lambda}\left[-\frac{1}{\lambda} e^{-\lambda x}\right]_0^\infty = \frac{2}{\lambda^2}$

$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

standard deviation of $X = \sqrt{\mathrm{Var}(X)} = \frac{1}{\lambda}$ $\square$

3b. $P(a < X \leq b) = \int_a^b \lambda e^{-\lambda x}\, dx = \left[-e^{-\lambda x}\right]_a^b$

$= e^{-\lambda a} - e^{-\lambda b}$ $\square$

3c. If ~~P(X~~ $(k-1) < X \leq k$, then

$P(k-1 < X \leq k) = e^{-\lambda(k-1)} - e^{-\lambda k}$ $\quad e^{-\lambda(k-1)}(1-e^{-\lambda})$

$= e^{-\lambda(k-1)}(1-e^{-\lambda})$

$= (e^{-\lambda})^{k-1}(1-e^{-\lambda})$

Let $p = 1 - e^{-\lambda}$,

$P(Y=k) = P(k-1 < X \leq k) = (1-p)^{k-1}(p)$ $\square$.

Probability mass function of a geometric distribution with parameter p is $(1-p)^{k-1} p$ $\therefore$ Y has a geometric distribution.

Parameter $= p = 1 - e^{-\lambda}$ $\square$

3d. If $X \sim \mathrm{Exp}(\lambda)$, we want to know the time elapsed before the first ~~"success"~~ accident happens.
If $X \sim \mathrm{Geo}(p)$, we want to know the number of trials before we get our first "success"

The result in (c) is obvious because if X is the time taken for a ~~per~~ particular bus to arrive, say, (note that in this case, then ~~the~~ buses arrive at a rate of λ per unit time), then Y must be the number of buses that have arrived before the particular bus of interest arrives.

4ai Let X be the final marks of the students.

$X \sim N(55, 10)$.

$P(X > 70) = P\left(Z > \frac{70-55}{10}\right) = P(Z > 1.5)$

$= 1 - 0.9332$

$= 0.0668$ ☐

4aii $P(X < 35) = P(Z < -2) = P(Z > 2)$

$= 1 - 0.9772$

$= 0.0228$ ☐

4aiii $P(60 \leq X < 70) = P(0.5 \leq Z \leq 1.5)$

$= 0.9332 - 0.6915$

$= 0.242$ ☐

4b Since no student scored above 70%, the sample mean, $\bar{x} < 70$.
Also since the expected proportion of students to score less than 35% is 0.0228, we can expect 0.0228×25 of the cohort to score less than 35%.

$\Rightarrow 0.0228 \times 25 = 0.57 < 1$

~~we can say~~
Assume that no student scores below 35%. $\therefore 35 < \bar{x} < 70$.

$H_0 : \mu = 55$.
$H_1 : \mu < 55$

Test statistic, $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \frac{\bar{X}-55}{10/5} = \frac{\bar{X}-55}{2}$

At 95% level, critical value $z_{0.05} = -1.6449$

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we accept H_0 if $Z > -1.6449$ and conclude that the test is not too difficult.

For $Z > -1.6449$, $\frac{\bar{x}-55}{2} > -1.6449$

$$\bar{x} > 51.7102$$

Since approx. 0.242 of the students score between 60% and 70%, number of students in the cohort scoring between 60% and ~~7%~~ 70%

$= 0.242 \times 25$

$= 6.05$

$\approx 6.$

$\therefore$ $25 - 6 = 19$ students score between 35% and 60%.

Expected mean score for students scoring between 60% and 70% = 65%.

Expected mean score for students scoring between 35% and 60%

$= (60-35)/2 + 35$

$= 47.5$ %.

$\therefore$ Expected value of $\bar{x} = \frac{65 \times 6 + 47.5 \times 19}{25} = 51.7.$

Since $\bar{x} \not> 51.7102$, we reject H_0 and conclude that the test is too difficult. □

4c It is reasonable to expect the scores to be normally distributed because by the Central Limit Theorem, if the sample ~~space~~ ~~is~~ size is large, then the scores will tend to a normal distribution. □

5a $\bar{x}_1 = \frac{1}{7}(94 + 197 + 16 + 38 + 99 + 141 + 23) = \frac{608}{7}$

$= 86.86$ □.

$\bar{x}_2 = \frac{1}{9}(52 + 104 + 146 + 10 + 50 + 31 + 40 + 27 + 46) = \frac{506}{9}$

$= 56.22$ □

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$s_1^2 = \frac{1}{7-1}\left[\sum x_i^2 - 7(\bar{x}_1)^2\right] = \frac{1}{6}\left[\frac{187228}{7}\right]$

$= 4457.8$ $\square$

$s_2^2 = \frac{1}{9-1}\left[\sum x_i^2 - 9(\bar{x}_2)^2\right] = \frac{1}{8}\left[\frac{129542}{9}\right] = 1799.2$ $\square$

5b. $H_0 : \sigma_1^2/\sigma_2^2 = 1$

$H_1 : \sigma_1^2/\sigma_2^2 \neq 1$

Test statistic $F = \frac{S_1^2}{S_2^2}$

$F \sim F_{6,8}$.

At 95% level, upper ~~lower~~ 2.5% point of $F_{6,8} = 4.652$.

lower 2.5% of $F_{6,8} = \frac{1}{5.6} = 0.1786$

realised value of $F = \frac{4457.8}{1799.2} = 2.478$

This value lies between 0.1786 and 4.652

$\therefore$ we accept H_0 and conclude that $\sigma_1^2 = \sigma_2^2$ $\square$

5c. ~~pot~~ pooled estimator, $s_p = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}$

$= \frac{6 \times 4457.8 + 8 \times 1799.2}{7+9-2}$

$= 2938.6$

At 95% level, critical values of t_{14} distribution $= \pm 2.145$

95% Confidence interval for $\mu_1 - \mu_2$ is

$\bar{x}_1 - \bar{x}_2 \pm \left(2.145 \times s_p\sqrt{\frac{1}{7}+\frac{1}{9}}\right)$

$= 86.86 - 56.22 \pm \left(2.145 \times \sqrt{\frac{16}{63}(2938.6)}\right)$

$= (-27.96, 89.23)$ $\square$

5d. The new treatment is not much more effective than the old one because the confidence interval contains 0, which implies $\mu_1 = \mu_2$. Also, we have accepted the hypothesis $\sigma_1^2 = \sigma_2^2$, meaning the variances of both treatments (new & old) are similar.

We have assumed that ~~$\bar{x}_1$ and $\bar{x}_2$~~ the survival times from both groups of mice ~~follo~~ have a normal distribution, and are independent from one another.

These assumptions are fine, since the mice are all different and how one mouse reacts to the treatment has no effect ~~of~~ on another mouse. $\square$

6a $S(\beta_0, \beta_1) = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$

$$\frac{dS}{d\beta_0} = -2 \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)$$

$$\frac{dS}{d\beta_0} = 0 \Rightarrow \sum_{i=1}^{n} y_i - \beta_0 - \beta_1 x_i = 0$$

$$n\bar{y} - n\beta_0 - n\beta_1 \bar{x} = 0$$

$$\hat{\beta}_0 = \bar{y} - \beta_1 \bar{x}. \quad \square$$

When $\beta_1 = 0$, $\hat{\beta}_0 = \bar{y}$ $\square$

6b. Suppose $T_1, T_2, \ldots, T_n$ are the possible estimators of θ. In order to choose the best possible estimator, we must find the mean-squared error $E[(T_i - \theta)^2]$, where $E(-)$ represents the expectation.

If we want an ~~a ub~~ unbiased estimator, then the one that gives the smallest mean squared error would be appropriate. However, taking an estimator with a small amount of bias might yield something with a smaller variance than the best unbiased estimator. Hence the bias estimator might be more suitable. $\square$